Positive solutions of singularly perturbed nonlinear elliptic problem on Riemannian manifolds with boundary

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Abstract

Let (M, g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ with a smooth boundary ∂M. We consider the problem

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, \; u > 0 \text{ on } M, \; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M$$

where ν is an exterior normal to ∂M.

The number of solutions of this problem depends on the topological properties of the manifold. In particular we consider the Lusternik Schnirelmann category of the boundary.

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1 Introduction

Let (M, g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ with a smooth boundary ∂M, that is ∂M is the union of a finite number of connected, smooth, boundaryless, submanifold of M of dimension $n-1$. Here g denotes the Riemannian metric tensor. By Nash theorem we can consider (M, g) embedded as a regular submanifold embedded in $\mathbb{R}^N$. We are interested in finding solutions $u \in H^1_g(M)$ of the following singularly perturbed nonlinear elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, & u > 0 \quad \text{on } M \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M \end{cases} \qquad (P)$$

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for $2 < p < 2^* = \frac{2N}{N-2}$, where ν is the external normal to ∂M.

Here $H^1_g(M) = \left\{ u : M \to \mathbb{R} : \int_M |\nabla_g u|^2 + u^2 d\mu_g < \infty \right\}$ where μ_g denotes the volume form on M associated to g.

Above type of equations have been extensively studied when M is a flat bounded domain $\Omega \subset \mathbb{R}^N$. We recall some classical result about the Neumann problem in Ω. In [16, 18, 19], Lin, Ni and Takagi established the existence of least-energy solution to (P) and showed that for ε small enough the least energy solution has a boundary spike. Later, in [11, 21] it was proved that for any stable critical point of the mean curvature of the boundary it is possible to construct single boundary spike layer solutions, while in [12, 15, 22] the authors construct multiple boundary spike solutions. Finally, in [9, 13] the authors proved that for any integer K there exists a boundary K-peaks solutions.

For which concerns the probem (P) on a manifold M, with boundary and without boundary, Byeon and Park [7] showed that the mountain pass solution u_ε has a spike layer.

A lot of works are devoted to show the influence of the topology of Ω on the number of solutions of the Dirichlet problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2} u, u > 0 & \text{on } \Omega \subset \mathbb{R}^N; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

when Ω is a flat subset of $\mathbb{R}^N$. We limit to cite [1, 2, 3, 5, 6, 7, 8].

Recently there have been some results on the effect of the topology of the manifold M on the number of solutions of the equation $-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$ on a manifold M without boundary. In [4] the authors proved that, if M has a rich topology, the equation has multiple solutions. More precisely they show that this equation has at least $\operatorname{cat}(M) + 1$ positive nontrivial solutions for ε small enough. Here $\operatorname{cat}(M)$ is the Lusternik-Schnirelmann category of M. In [20] there is the same result for a more general nonlinearity. Furthermore in [14] it was shown that the number of solution is influenced by the topology of a suitable subset of M depending on the geometry of M.

Our result concerns problem (P) on a manifold M with $\partial M \neq \emptyset$. In this case we show that the topology of the boundary ∂M influences the number of solutions, as follows.

Theorem 1. *For ε small enough the problem (P) has at least* $\operatorname{cat}(\partial M) + 1$ *non constant distinct solutions.*

The paper is organized as follows. In Section 2 we introduce some notions and notations. In Section 3 we sketch the proof of the main result. The details of the proof are in sections 4-7.

2 Preliminaries

We consider the C^2 functional defined on $H^1_g(M)$

$$J_\varepsilon(u) = \frac{1}{\varepsilon^N}\int_M \left(\frac{1}{2}\varepsilon^2|\nabla_g u|^2 + \frac{1}{2}|u|^2 - \frac{1}{p}|u^+|^p\right) d\mu_g. \tag{2}$$

where $u^+(x) = \max\{u(x), 0\}$. It is well known that the critical points of $J_\varepsilon(u)$ constrained on the associated C^2 Nehari manifold

$$\mathcal{N}_\varepsilon = \left\{u \in H^1_g \smallsetminus \{0\} \ : \ J'_\varepsilon(u)u = 0\right\} \tag{3}$$

are non trivial solution of problem (P).

Let $\mathbb{R}^n_+ = \{x = (\bar{x}, x_n) : \bar{x} \in \mathbb{R}^{n-1}, x_n \geq 0\}$. It is known that there exists a least energy solution $V \in H^1(\mathbb{R}^n_+)$ of the equation

$$\begin{cases} -\Delta V + V = |V|^{p-2}V, \ V > 0 \quad \text{on } \mathbb{R}^n_+ \\ \frac{\partial V}{\partial x_n}|_{(\bar{x},0)} = 0. \end{cases} \tag{4}$$

Moreover V is radially symmetric and $|D^\alpha V(x)| \leq c\exp(-\mu\,|x|)$ with $|\alpha| \leq 2$, and c, μ positive constants.

If V is a solution, also $V(x+y)$ with $y = (\bar{y}, 0)$ is a solution, $V_\varepsilon(x) = V\left(\frac{x}{\varepsilon}\right)$ is a solution of

$$\begin{cases} -\varepsilon^2\Delta V_\varepsilon + V_\varepsilon = |V_\varepsilon|^{p-2}V_\varepsilon \quad \text{on } \mathbb{R}^n_+ \\ \frac{\partial V_\varepsilon}{\partial x_n}|_{(\bar{x},0)} = 0. \end{cases} \tag{5}$$

We put

$$m_e^+ = \inf\left\{E^+(v) : v \in \mathcal{N}(E^+)\right\} \text{ and } m_e = \inf\left\{E(v) : v \in \mathcal{N}(E)\right\}, \tag{6}$$

where

$$\begin{aligned} E^+(v) &= \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla v|^2 + \frac{1}{2}|v|^2 - \frac{1}{p}|v^+|^p dx; \\ E(v) &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla v|^2 + \frac{1}{2}|v|^2 - \frac{1}{p}|v^+|^p dx. \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}(E^+) &= \left\{v \in H^1(\mathbb{R}^n_+) \smallsetminus \{0\} \ : \ E^+(v)v = 0\right\}; \\ \mathcal{N}(E) &= \left\{v \in H^1(\mathbb{R}^n) \smallsetminus \{0\} \ : \ E(v)v = 0\right\}. \end{aligned}$$

It holds

$$m_e = 2m_e^+,$$

and

$$m_e^+ = E^+(V) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(S_e^+\right)^{\frac{p}{p-2}} \text{ where } S_e^+ = \inf\left\{\frac{||v||^2_{H^1(\mathbb{R}^n_+)}}{||v||^2_{L^p(\mathbb{R}^n_+)}}, v \neq 0\right\}.$$

Remark 2. On the tangent bundle of any compact Riemannian manifold $\mathcal{M}$ it is defined the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ which is of class C^∞. Moreover there exists a constant $R > 0$ and a finite number of $x_i \in \mathcal{M}$ such that $\mathcal{M} = \cup_{i=1}^l B_g(x_i, R)$ and $\exp_{x_i} : B(0,R) \to B_g(x_i, R)$ is a diffeormophism for all i.

By choosing an orthogonal coordinate system $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$ and identifying $T_{x_0}\mathcal{M}$ with $\mathbb{R}^n$ for $x_0 \in \mathcal{M}$ we can define by the exponential map the so called normal coordinates. For $x_0 \in \mathcal{M}$, g_{x_0} denotes the metric read through the normal coordinates. In particular, we have $g_{x_0}(0) = \mathrm{Id}$. We set $|g_{x_0}(y)| = \det\left(g_{x_0}(y)\right)_{ij}$ and $g_{x_0}^{ij}(y) = \left(\left(g_{x_0}(y)\right)_{ij}\right)^{-1}$.

Remark 3. If q belongs to the boundary ∂M, let $\bar{y} = (y_1, \ldots, y_{n-1})$ be Riemannian normal coordinates on the $n-1$ manifold ∂M at the point q. For a point $\xi \in M$ close to q, there exists a unique $\bar{\xi} \in \partial M$ such that $d_g(\xi, \partial M) = d_g(\xi, \bar{\xi})$. We set $\bar{y}(\xi) \in \mathbb{R}^{n-1}$ the normal coordinates for $\bar{\xi}$ and $y_n(\xi) = d_g(\xi, \partial M)$. Then we define a chart $\psi_q^\partial : \mathbb{R}^n_+ \to M$ such that $(\bar{y}(\xi), y_n(\xi)) = \left(\psi_q^\partial\right)^{-1}(\xi)$. These coordinates are called *Fermi coordinates* at $q \in \partial M$. The Riemannian metric $g_q(\bar{y}, y_n)$ read through the Fermi coordinates satisfies $g_q(0) = \mathrm{Id}$.

In the following we choose $\rho > 0$ such that in the subset $(\partial M)_\rho := \{x \in M : d_g(x, \partial M) < \rho\}$ the Fermi coordinates are well defined. Moreover we choose ρ small enough such that 3ρ is smaller than the radius $\rho(\partial M)$ of topological invariance of ∂M, defined below.

Definition 1. *The radius of topological invariance $\rho(\mathcal{M})$ of $\mathcal{M} \subset \mathbb{R}^N$ is*

$$\rho(\mathcal{M}) := \sup\left\{\rho > 0 : \operatorname{cat}\left((\mathcal{M})_\rho\right) = \operatorname{cat}(\mathcal{M})\right\}$$

where

$$(\mathcal{M})_\rho := \left\{x \in \mathbb{R}^N : d(x, \mathcal{M}) < \rho\right\}$$

Fixed ρ, using Remark 2, we can choose R_M such that $\cup_{i=1}^l B_g(x_i, R_M)$ covers $M \smallsetminus (\partial M)_\rho$, and $R_M < \rho$. We note by d_g^∂ and $\exp^\partial$ respectively the geodesic distance and the exponential map on by ∂M. By compactness of ∂M, there is an R^∂ and a finite number of points $q_i \in \partial M$, $i = 1, \ldots, k$ such that

$$I_{q_i}(R^\partial, \rho) := \left\{x \in M,\ d_g(x, \partial M) = d_g(x, \bar{\xi}) < \rho,\ d_g^\partial(q_i, \bar{\xi}) < R^\partial\right\}$$

form a covering of $(\partial M)_\rho$ and on every I_{q_i} the fermi coordinates are well defined. In the following we can choose without loss of generality, $R = \min\left\{R^\partial, R_M\right\} < \rho$.

3 Main tools for the proof

Using the notation of the previous section we can state our main result more precisely.

Theorem 4. *There exists $\delta_0 \in (0, m_e^+)$ and $\varepsilon_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_0)$ the functional J_ε has at least $\operatorname{cat}(\partial M)$ critical points $u \in \mathcal{N}_\varepsilon \subset H_g^1(M)$ satisfying $J_\varepsilon(u) < m_e^+ + \delta$ and at least a critical point with $m_e^+ + \delta \leq J_\varepsilon(u) \leq c$.*

We recall the definition of Lusternik Schnirelmann category.

Definition 2. *Let M a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\operatorname{cat}_M A = k$) if A is covered by k closed sets A_j, $j = 1, \dots, k$, which are contractible in M, and k is the minimum integer with this property.*

Remark 5. Let M_1 and M_2 be topological spaces. If $g_1 : M_1 \to M_2$ and $g_2 : M_2 \to M_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on M_1, then $\operatorname{cat} M_1 \leq \operatorname{cat} M_2$. For the proof see [5].

We recall the following classical result (see for example [6]).

Theorem 6. *Let J be a $C^{1,1}$ real functional on a complete $C^{1,1}$ manifold $\mathcal{N}$. If J is bounded from below and satisfies the* Palais Smale *condition then has at least* $\operatorname{cat}(J^d)$ *critical point in J^d where $J^d = \{u \in \mathcal{N} \ : \ J(u) < d\}$. Moreover if $\mathcal{N}$ is contractible and* $\operatorname{cat} J^d > 1$*, there exists at least one critical point $u \notin J^d$*

Applying the first claim of Theorem 6 to the functional J_ε on the manifold $\mathcal{N}_\varepsilon$ we obtain $\operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ critical points of J_ε. By the following Lemma we give an estimate of $\operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ through the topological properties of the boundary of M.

Lemma 7. *For δ and ε small enough we have* $\operatorname{cat}(\partial M) \leq \operatorname{cat} \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$.

We are able to obtain the proof of this lemma building two suitable maps. To this aim we recall that by Nash embedding theorem [17] we may assume that M is embedded in a Euclidean space $\mathbb{R}^N$.

Hence the lemma follows by building a map $\Phi_\varepsilon : \partial M \to \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ and a map $\beta : \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \to (\partial M)_\rho$ with $0 < \rho < \rho(\partial M)$ such that $\beta \circ \Phi_\varepsilon : \partial M \to (\partial M)_\rho$ is homotopic to the identity on ∂M (see sections 4,5,6). Then by the properties of the category we get $\mathrm{cat}(\partial M) \le \mathrm{cat}\, \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$.

To finish the proof of Theorem 4 we build a set T_ε (Section 7) such that

$$\Phi_\varepsilon(\partial M) \subset T_\varepsilon \subset \mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$$

for a bounded constant $c_\varepsilon \le c$, and such that T_ε is a contractible set in $\mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$ containing only positive functions. Since $1 < \mathrm{cat}(\partial M) \le \mathrm{cat}\,(\Phi_\varepsilon(\partial M))$ by the same argument of Theorem 6 there exists a critical point $\bar{u}$ of J_ε in $\mathcal{N}_\varepsilon$ such that $m_e^+ + \delta \le J_\varepsilon(\bar{u}) \le c_\varepsilon$.

It remains to show that the critical points we have found are non-constant functions. This follows immediately from the fact that the only constant function on the Nehari manifold $\mathcal{N}_\varepsilon$ is the function $\bar{v}(x) \equiv 1$, for which

$$J_\varepsilon(\bar{v}) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{\mu_g(M)}{\varepsilon^n} \to \infty \text{ as } \varepsilon \to 0.$$

Hence the constant solution is excluded because c_ε is bounded.

3.1 Notation

We will use the following notation

- $||u||_g = ||u||_{H_g^1} = \int_M |\nabla_g u|^2 + |u|^2 d\mu_g, \ \ |u|_{p,g}^p = \int_M |u|^p d\mu_g;$
- $|||u|||_\varepsilon = |||u|||_{\varepsilon,M} = \frac{1}{\varepsilon^n} \int_M \varepsilon^2 |\nabla_g u|^2 + |u|^2 d\mu_g, \ \ |u|_{p,\varepsilon}^p = \frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g;$
- $|u|_p^p = \int_{\mathbb{R}^n} |u|^p dx;$
- If $A, B \subset \mathbb{R}^n$, then $A \Delta B := A \smallsetminus B \cup B \smallsetminus A$.
- d_g is the geodesic distance on M, and d_g^∂ is the geodesic distance on ∂M.
- $\exp^\partial$ is the exponential map on ∂M.
- $I_q(R,\rho) = \{\chi \in M \ : \ d_g(\chi, \partial M) < \rho, \ d_g^\partial(\bar{\chi}, q) < R\}$, where $\bar{\chi} \in \partial M$ is the unique point such that $d_g(\chi, \bar{\chi}) = d_g(\chi, \partial M)$.
- $B(x,R) \subset \mathbb{R}^n$ is the ball centered in x of radius R.
- $B_{n-1}(x,R) \subset \mathbb{R}^{n-1}$ is the $n-1$ ball centered in x of radius R.

4 The map Φ_ε

Let us define $\chi_R : \mathbb{R}^+ \to \mathbb{R}^+$ a smooth cut off function such that $\chi_R(t) \equiv 1$ if $0 \le t \le R/2$, $\chi_R(t) \equiv 0$ if $R \le t$, and $|\chi_R'(t)| \le 2/R$ for all t. Fixed a point $q \in \partial M$ and $\varepsilon > 0$, let us define on M the function $Z_{\varepsilon,q}(\xi)$ as

$$Z_{\varepsilon,q}(\xi) = \begin{cases} V_\varepsilon\left(y(\xi)\right)\chi_R\left(|\bar{y}(\xi)|\right)\chi_\rho\left(y_n(\xi)\right) & \text{if } \xi \in I_q \\ \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where

$$I_q(R,\rho) = I_q = \left\{\xi \in M : y_n = d_g(\xi, \partial M) < \rho \text{ and } |\bar{y}| = d_g^\partial\left(\exp_q^\partial(\bar{y}(\xi)), q\right) < R\right\}.$$

Here $y(\xi) = (\bar{y}(\xi), y_n(\xi)) = \left(\psi_q^\partial\right)^{-1}(\xi)$ are the Fermi coordinates at $q \in \partial M$ and $\exp_q^\partial : T_q(\partial M) \to \partial M$, is the exponential map on ∂M.

For each $\varepsilon > 0$ we can define a positive number $t_\varepsilon(Z_{\varepsilon,q})$ such that $t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q} \in H_g^1(M) \cap \mathcal{N}_\varepsilon$. Namely, $t_\varepsilon(Z_{\varepsilon,q})$ turns out to verify

$$t_\varepsilon(Z_{\varepsilon,q}) = \left(\frac{|||Z_{\varepsilon,q}|||_\varepsilon^2}{|Z_{\varepsilon,q}|_{p,\varepsilon}^p}\right)^{\frac{1}{p-2}}. \tag{8}$$

Thus we can define a function $\Phi_\varepsilon : \partial M \to \mathcal{N}_\varepsilon$, $\Phi_\varepsilon(q) = t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}$

Proposition 8. *For any $\varepsilon > 0$ the application $\Phi_\varepsilon : \partial M \to \mathcal{N}_\varepsilon$ is continuous. Moreover, for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, if $\varepsilon < \varepsilon_0$ then*

$$\Phi_\varepsilon(q) \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta} \text{ for all } q \in \partial M$$

Proof. Fixed $\varepsilon > 0$, by the continuity of $u \to t_\varepsilon(u)$ on $H_g^1(M)$ it is enough to prove that for any sequence $\{q_k\} \subset \partial M$ convergent to q we have

$$\lim_{k\to\infty} ||Z_{\varepsilon,q_k} - Z_{\varepsilon,q}||_{H_g^1} = 0.$$

Since q_k converges to q, we have $\mu_g(I_{q_k}\Delta I_q) \to 0$ as $k \to \infty$, then we have

$$\int_{I_{q_k}\Delta I_q} |Z_{\varepsilon,q_k} - Z_{\varepsilon,q}|^2\, d\mu_g \to 0 \text{ as } k \to \infty.$$

Now, setting $\eta_k(\bar{y}, y_n) = \left(\psi_{q_k}^\partial\right)^{-1}\left(\psi_q^\partial(\bar{y}, y_n)\right)$ and $A_k = \left(\psi_q^\partial\right)^{-1}(I_{q_k} \cap I_q)$ we can write

$$\int_{I_{q_k}\cap I_q} |Z_{\varepsilon,q_k}(x) - Z_{\varepsilon,q}(x)|^2 \, d\mu_g =$$

$$\int_{A_k} \Big| V_\varepsilon(\eta_k(\bar{y}, y_n))\chi_R\left(|\pi_{\mathbb{R}^{n-1}}\eta_k(\bar{y}, y_n)|\right)\chi_\rho\left(d_g(q_k, \partial M)\right) -$$
$$- V_\varepsilon\left(\bar{y}, y_n\right)\chi_R\left(|\bar{y}|\right)\chi_\rho\left(d_g(q, \partial M)\right)\Big|^2 |g_q(\bar{y}, y_n)|^{1/2} \, d\bar{y}dy_n \leq$$
$$\leq c \int_{A_k} |\eta_k(\bar{y}, y_n) - (\bar{y}, y_n)|^2 \, d\bar{y}dy_n$$

for a suitable constant c coming from the mean value theorem applied to $V_\varepsilon, \chi_\rho, \chi_R$. By the definition of η_k and the smoothness of the exponential map we get

$$||Z_{\varepsilon,q_k} - Z_{\varepsilon,q}||_{L^2_g} \to 0 \text{ as } k \to \infty.$$

A similar argument can be used to show that $||\nabla_g Z_{\varepsilon,q_k} - \nabla_g Z_{\varepsilon,q}||_{L^2_g} \to 0$ as $k \to \infty$.

To prove the second statement of the theorem we first show that the following limits hold uniformly with respect to $q \in \partial M$.

$$\lim_{\varepsilon\to 0} ||Z_{\varepsilon,q}||^2_{2,\varepsilon} = \int_{\mathbb{R}^n_+} V^2(y)dy \tag{9}$$

$$\lim_{\varepsilon\to 0} ||Z_{\varepsilon,q}||^p_{p,\varepsilon} = \int_{\mathbb{R}^n_+} V^p(y)dy \tag{10}$$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \, ||\nabla Z_{\varepsilon,q}||^2_{2,\varepsilon} = \int_{\mathbb{R}^n_+} |\nabla V|^2 \, (y)dy \tag{11}$$

where $||u||_{q,\varepsilon} = \frac{1}{\varepsilon^n} ||u||_{L^q}$. For (9) we have

$$\frac{1}{\varepsilon^n}\int_M |Z_{\varepsilon,q}(x)|^2 \, d\mu_g =$$
$$= \frac{1}{\varepsilon^n}\int_{|\bar{y}|<R, 0<y_n<\rho} V^2_\varepsilon(\bar{y}, y_n)\chi^2_R\left(|\bar{y}|\right)\chi^2_\rho\left(y_n\right)|g_q(\bar{y}, y_n)|^{1/2} \, d\bar{y}dy_n =$$
$$= \int_{|\bar{z}|<R/\varepsilon, 0<z_n<\rho/\varepsilon} V^2(\bar{z}, z_n)\chi^2_{R/\varepsilon}\left(|\bar{z}|\right)\chi^2_{\rho/\varepsilon}\left(z_n\right)|g_q(\varepsilon(\bar{z}, z_n))|^{1/2} \, d\bar{z}dz_n =$$
$$= \int_{B_K} V^2(\bar{z}, z_n)\chi^2_{R/\varepsilon}\left(|\bar{z}|\right)\chi^2_{\rho/\varepsilon}\left(z_n\right)|g_q(\varepsilon(\bar{z}, z_n))|^{1/2} \, d\bar{z}dz_n +$$
$$+ \int_{\mathbb{R}^n_+\smallsetminus B_K} V^2(\bar{z}, z_n)\chi^2_{R/\varepsilon}\left(|\bar{z}|\right)\chi^2_{\rho/\varepsilon}\left(z_n\right)|g_q(\varepsilon(\bar{z}, z_n))|^{1/2} \, d\bar{z}dz_n,$$

where $B_k = B(0,K) \cap \{z_n > 0\}$. It is easy to see that the second addendum vanishes when $K \to \infty$. With respect to the first addendum, fixed K large enough, by compactness of manifold M and regularity of the exponential map and of the Riemannian metric g we have, for $\varepsilon \to 0$,

$$\int_{B_K} V^2(\bar{z}, z_n)\chi^2_{R/\varepsilon}(|\bar{z}|)\,\chi^2_{\rho/\varepsilon}(z_n)\left|g_{\psi_q^\partial}(\varepsilon(\bar{z}, z_n))\right|^{1/2} d\bar{z}dz_n \to \int_{B_K} V^2(y)dy$$

uniformly with respect to $q \in \partial M$. So we prove (9). In the same way we can prove (10) and (11).

At this point we observe that

$$J_\varepsilon(t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}) = \left(\frac{1}{2} - \frac{1}{p}\right)[t_\varepsilon(Z_{\varepsilon,q})]^p\,||Z_{\varepsilon,q}||^p_{\varepsilon,p}\,.$$

By definition of $t_\varepsilon(Z_{\varepsilon,q})$ and by (9), (10) and (11) we have that $t_\varepsilon(Z_{\varepsilon,q}) \to 1$ as $\varepsilon \to 0$, uniformly with respect to $q \in \partial M$. Concluding we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}) = \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^n_+} V^p(y)dy = m_e^+ \tag{12}$$

uniformly with respect to $q \in \partial M$. □

Remark 9. By Proposition 8, given δ, we have that $\mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \neq \emptyset$ for ε small enough. Moreover let

$$m_\varepsilon := \inf\{J_\varepsilon(u) \,:\, u \in \mathcal{N}_\varepsilon\}\,.$$

At this point we have

$$\limsup_{\varepsilon\to 0} m_\varepsilon \le m_e^+.$$

5 Concentration properties

In this section we will show a property of concentration of the functions $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ when ε and δ are sufficiently small. This concentration property will be crucial to verify that the barycenter $\beta(u)$ (see Section 6) of the functions $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ is close to the boundary ∂M.

For any $\varepsilon > 0$ we can construct a finite closed partition $\mathcal{P}^\varepsilon = \left\{P_j^\varepsilon\right\}_{j\in\Lambda_\varepsilon}$ of M such that

- P_j^ε is closed for every j;
- $P_j^\varepsilon \cap P_k^\varepsilon \subset \partial P_j^\varepsilon \cap \partial P_k^\varepsilon$ for $j \neq k$;

- $K_1\varepsilon \leq d_j^\varepsilon \leq K_2\varepsilon$, where d_j^ε is the diameter of P_j^ε;
- $c_1\varepsilon^n \leq \mu_g\left(P_j^\varepsilon\right) \leq c_2\varepsilon^n$;
- for any j there exists an open set $I_j^\varepsilon \supset P_j^\varepsilon$ such that, if $P_j^\varepsilon \cap \partial M = \emptyset$, then $d_g\left(I_j^\varepsilon, \partial M\right) > K\varepsilon/2$, while, if $P_j^\varepsilon \cap \partial M \neq \emptyset$, then $I_j^\varepsilon \subset \left\{x \in M \,:\, d_g(x, \partial M) \leq \frac{3}{2}K\varepsilon\right\}$;
- there exists a finite number $\nu(M) \in \mathbb{N}$ such that every $x \in M$ is contained in at most $\nu(M)$ sets I_j^ε, where $\nu(M)$ does not depends on ε.

By compactness of M such a partition exists, at least for small ε. In the following we will choose always $\varepsilon_0(\delta)$ sufficiently small in order to have this partition.

Lemma 10. *There exists a constant $\gamma > 0$ such that, for any fixed $\delta > 0$ and for any $\varepsilon \in (0, \varepsilon_0(\delta))$, where $\varepsilon_0(\delta)$ is as in Proposition 8, given any partition $\mathcal{P}^\varepsilon$ of M as above, and any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta}$, there exists a set $P_j^\varepsilon \subset \mathcal{P}^\varepsilon$ such that*

$$\frac{1}{\varepsilon^n}\int_{P_j^\varepsilon} |u^+|^p d\mu_g \geq \gamma > 0.$$

Proof. By Remark 9 we have that $\mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta} \neq \emptyset$. For any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta}$ we denote by u_j^+ the restriction of u^+ to the set P_j^ε. Then we can write

$$\frac{1}{\varepsilon^n}\int_M \left(\varepsilon^2|\nabla_g u|^2 + u^2\right) d\mu_g = \frac{1}{\varepsilon^n}\int_M (u^+)^p d\mu_g = \frac{1}{\varepsilon^n}\sum_j \int_M (u_j^+)^p d\mu_g =$$

$$= \sum_j \frac{|u_j^+|_p^{p-2}}{\varepsilon^{\frac{n(p-2)}{p}}}\frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}} \leq \max_j \left\{\frac{|u_j^+|_p^{p-2}}{\varepsilon^{\frac{n(p-2)}{p}}}\right\}\sum_j \frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}}.$$

We define the functions $\tilde{u}_j$ by using a smooth real cutoff function $\chi_\varepsilon^j : M \to [0,1]$ such that $|\nabla_g \chi_\varepsilon^j| \leq \frac{K}{\varepsilon}$ for some constant K and, if $P_j^\varepsilon \cap \partial M = \emptyset$, then $\chi_\varepsilon^j = 1$ for $x \in P_j^\varepsilon$ and $\chi_\varepsilon^j = 0$ for $x \in M \smallsetminus I_j^\varepsilon$, while if $P_j^\varepsilon \cap \partial M \neq \emptyset$, then $\chi_\varepsilon^j = 1$ for $x \in P_j^\varepsilon$ and $\chi_\varepsilon^j = 0$ for $M \smallsetminus \bar{I}_j^\varepsilon$ and $x \in \partial I_j^\varepsilon \cap (M \smallsetminus \partial M)$. So we define

$$\tilde{u}_j(x) = u^+(x)\chi_\varepsilon^j(x).$$

It holds $\tilde{u}_j \in H^1_g(M)$, hence using Sobolev inequalities there exists a positive constant C such that, for any j,

$$\frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}} \leq \frac{|\tilde{u}_j|_p^2}{\varepsilon^{\frac{2n}{p}}} \leq C|||\tilde{u}_j|||_\varepsilon^2 = C|||\tilde{u}_j|||^2_{\varepsilon,P_j^\varepsilon} + C|||\tilde{u}_j|||^2_{\varepsilon,I_j^\varepsilon \smallsetminus P_j^\varepsilon}.$$

Moreover

$$\begin{aligned}
\int_{I_j^\varepsilon \smallsetminus P_j^\varepsilon} |\tilde{u}_j|^2 \, d\mu_g &\leq \int_{I_j^\varepsilon \smallsetminus P_j^\varepsilon} \left|u^+\right|^2 d\mu_g; \\
\int_{I_j^\varepsilon \smallsetminus P_j^\varepsilon} \varepsilon^2 \left|\nabla \tilde{u}_j\right|^2 d\mu_g &\leq \int_{I_j^\varepsilon \smallsetminus P_j^\varepsilon} (\varepsilon^2 \left|\nabla u^+\right|^2 + K^2 \left|u^+\right|^2) d\mu_g.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\sum_j \frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}} &\leq C \sum_j \left|\left|\left|u^+\right|\right|\right|_\varepsilon^2 + C(K^2+1)\nu(M) \left|\left|\left|u^+\right|\right|\right|_\varepsilon^2 \leq \\
&\leq C(K^2+2)\nu(M) \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 \left|\nabla u\right|^2 + \left|u\right|^2) d\mu_g.
\end{aligned}$$

We can conclude that

$$\max_j \left\{ \left(\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} \left|u^+\right|^p d\mu_g \right)^{\frac{p-2}{p}} \right\} \geq \frac{1}{C(K^2+2)\nu(M)},$$

so the proof is complete. □

Remark 11. Let δ and ε fixed. For any $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon + 2\delta}$ there exists $u_\delta \in \mathcal{N}_\varepsilon$ such that

$$J_\varepsilon(u_\delta) < J_\varepsilon(u), \;\; |||u_\delta - u|||_\varepsilon < 4\sqrt{\delta};$$
$$\left| \left(J_{\varepsilon|\mathcal{N}_\varepsilon}\right)'(u_\delta)[\xi] \right| < \sqrt{\delta} \, |||\xi|||_\varepsilon \, .$$

This is simply the application of Ekeland variational principle (see [10]) to the functional J_ε on the manifold $\mathcal{N}_\varepsilon$.

Proposition 12. *For all $\eta \in (0,1)$ there exists a $\delta_0 < m_e^+$ such that for any $\delta \in (0, \delta_0)$ for any $\varepsilon \in (0, \varepsilon_0(\delta))$ (as in Prop. 8) and for any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta}$ we can find a point $q = q(u) \in \partial M$ for which*

$$\left(\frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon^n} \int_{I_q(\rho,R)} |u^+|^p d\mu_g \geq (1-\eta) m_e^+$$

where $I_q(\rho, R)$ is defined in the notation paragraph.

Proof. We prove this property for $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_e^+ + \delta} \cap J_\varepsilon^{m_\varepsilon + 2\delta}$. From the thesis for these functions follows that

$$m_\varepsilon \geq (1-\eta) m_e^+. \tag{13}$$

By (13) and by Remark 9 we have that

$$\lim_{\varepsilon \to 0} m_\varepsilon = m_e^+. \tag{14}$$

Thus $J_\varepsilon^{m_e^+ + \delta} \subset J_\varepsilon^{m_\varepsilon + 2\delta}$ for ε, δ small enough, and the general case is proved.

The proof is by contradiction. Hence we assume that there exists $\eta \in (0,1)$, two sequences of vanishing real numbers $\{\delta_k\}_k$ and $\{\varepsilon_k\}_k$ and a sequence of functions $\{u_k\}_k \subset \mathcal{N}_{\varepsilon_k} \cap J_{\varepsilon_k}^{m_e^+ + \delta_k} \cap J_{\varepsilon_k}^{m_{\varepsilon_k} + 2\delta_k}$ such that, for any $q \in \partial M$ it holds

$$\left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon_k^n} \int_{I_q(\rho, R)} |u_k^+|^p d\mu_g < (1-\eta) m_e^+. \tag{15}$$

By Remark 11 and by definition of $\mathcal{N}_{\varepsilon_k}$ we can assume

$$J'_{\varepsilon_k}(u_k)[\varphi] \leq \sqrt{\delta_k}\, |||\varphi|||_{\varepsilon_k} \text{ for all } \varphi \in H_g^1(M).$$

By Lemma 10 there exists a set $P_k^{\varepsilon_k} \in \mathcal{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^n} \int_{P_k^{\varepsilon_k}} |u_k^+|^p d\mu_g \geq \gamma > 0.$$

we have to examine two cases: either there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M \neq \emptyset$, or there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M = \emptyset$. For simplicity we write simply P_k for $P_{i_k}^{\varepsilon_{i_k}}$.

The case $P_k \cap \partial M \neq \emptyset$. We choose a point q_k interior to $P_k \cap \partial M$. We have the Fermi coordinates $\psi_{q_k}^\partial : B_{n-1}(0,R) \times [0,\rho] \to M$, $\psi_{q_k}^\partial(\bar{y}, y_n) = (\bar{x}, x_n) = x$. We consider the function $w_k : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$u_k(\psi_{q_k}^\partial(\bar{y}, y_n)) \chi_R(|\bar{y}|) \chi_\rho(y_n) = u_k(\psi_{q_k}^\partial(\varepsilon_k \bar{z}, \varepsilon z_n)) \chi_R(|\varepsilon_k \bar{z}|) \chi_\rho(\varepsilon z_n) = w_k(\bar{z}, z_n).$$

It is clear that $w_k \in H^1(\mathbb{R}_+^n)$ with $w_k(\bar{z}, z_n) = 0$ when $|\bar{z}| = 0, R/\varepsilon_k$ or $z_n = \rho/\varepsilon_k$. We now show some properties of the function w_k.

STEP1: *There exists a $w \in H^1(\mathbb{R}_+^n)$ such that the sequence w_k converges weakly in $H^1(\mathbb{R}_+^n)$ and strongly in $L_{loc}^p(\mathbb{R}_+^n)$*

We have the following inequality

$$\begin{aligned}
&\frac{1}{\varepsilon_k^n}\int_M |u_k|^2 d\mu_g \geq \\
&\geq \frac{1}{\varepsilon_k^n}\int_{B_{n-1}(0,R)\times[0,\rho]} |u_k\left(\psi^\partial_{q_k}(y)\right)|^2\chi_R^2\left(|\bar y|\right)\chi_\rho^2\left((y_n)\right)|g_{q_k}(y)|^{1/2}\,dy = \\
&= \int_{B_{n-1}(0,R/\varepsilon_k)\times[0,\rho/\varepsilon_k]} |w_k|^2\,|g_{q_k}(\varepsilon z)|^{1/2}\,dz \geq c|w_k|^2_{L^2(\mathbb{R}^n_+)}.
\end{aligned} \tag{16}$$

Where $z=\varepsilon y$ and $c>0$ is a suitable constant.

For simplicity we set $\tilde\chi(y)=\chi_R(\bar y)\chi_\rho(y_n)$ We have

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}|\nabla w_k|^2\,dx \leq \\
\leq\ & 2\int_{\mathbb{R}^n_+}\sum_i\left(\frac{\partial u_k}{\partial z_i}(\varepsilon_k z)\right)^2\tilde\chi^2(\varepsilon_k z)dz + 2\int_{\mathbb{R}^n_+}\sum_i u_k^2(\varepsilon_k z)\left(\frac{\partial\tilde\chi}{\partial z_i}(\varepsilon_k z)\right)^2 dz \\
=\ & I_1+I_2
\end{aligned}$$

By definition of $\tilde\chi$ and w_k we have

$$\begin{aligned}
&\frac{\varepsilon_k^2}{\varepsilon_k^n}\int_M|\nabla_g u_k|^2\,d\mu_g \geq \frac{\varepsilon_k^2}{\varepsilon_k^n}\int_{\psi^\partial_{q_k}(B_{n-1}(0,R)\times[0,\rho])}|\nabla_g u_k|^2\,d\mu_g = \\
=\ &\int_{B_{n-1}(0,R/\varepsilon_k)\times[0,\rho/\varepsilon_k]}\sum_{ij} g^{ij}_{q_k}\frac{\partial u_k}{\partial z_i}(\varepsilon_k z)\frac{\partial u_k}{\partial z_j}(\varepsilon_k z)\,|g_{q_k}(\varepsilon z)|^{1/2}\,dz \geq cI_1.
\end{aligned} \tag{17}$$

where c depends only on the Riemannian manifold M. In a similar way we have

$$I_2 \leq \frac{c\varepsilon_k^2}{R^2\rho^2\varepsilon_k^n}\int_M|u_k|^2\,d\mu_g. \tag{18}$$

By (16), (17) and (18) we get that $||w_k||_{H^1(\mathbb{R}^n_+)}$ is bounded. Then we have the claim.

Step2: *The limit function w is a weak solution of*

$$\begin{cases} -\Delta w + w = (w^+)^{p-1} & \text{in } \mathbb{R}^n_+; \\ \frac{\partial w}{\partial\nu}=0 & \text{for } y=(\bar y,0); \end{cases}$$

Firstly for any $\varphi\in C_0^\infty(\mathbb{R}^n_+)$ we define on the manifold M the function $\tilde\varphi_k(x):=\varphi\left(\frac{1}{\varepsilon_k}\left(\psi^\partial_{q_k}\right)^{-1}(x)\right)$. We have that

$$\begin{aligned}
|||\tilde\varphi_k|||_{\varepsilon_k} &= \int_{\mathbb{R}^n_+}\left[\sum_{ij}g^{ij}_{q_k}(\varepsilon_k z)\frac{\partial\varphi}{\partial z_i}(z)\frac{\partial\varphi}{\partial z_j}(z)+|\varphi(z)|^2\right]|g_{q_k}(\varepsilon_k z)|^{1/2}dz \\
&\leq c||\varphi||^2_{H^1(\mathbb{R}^n_+)}
\end{aligned} \tag{19}$$

where c depends only on M.

We set

$$F_{\varepsilon_k}(v)=\int_{\mathbb{R}^n_+}\left[\sum_{ij}\frac{g^{ij}_{q_k}(\varepsilon_k z)}{2}\frac{\partial v}{\partial z_i}(z)\frac{\partial v}{\partial z_j}(z)+\frac{v^2(z)}{2}-\frac{|w_k^+(z)|^p}{p}\right]|g_{q_k}(\varepsilon_k z)|^{1/2}dz$$

so

$$\begin{aligned}&|F'_{\varepsilon_k}(w_k)[\varphi]|=\\&=\int_{\mathrm{supp}\varphi}\left[\sum_{ij}g^{ij}_{q_k}(\varepsilon_k z)\frac{\partial w_k}{\partial z_i}(z)\frac{\partial \varphi}{\partial z_j}(z)+\left(w_k(z)-(w_k^+(z))^{p-1}\right)\varphi(z)\right]|g_{q_k}(\varepsilon_k z)|^{1/2}.\end{aligned}$$

It is easy to verify that for $k=k(\varphi)$ large enough

$$|F'_{\varepsilon_k}(w_k)[\varphi]|=|J'_{\varepsilon_k}(u_k)[\tilde{\varphi}_k]|.$$

By Ekeland principle (Remark 11) and by (19) we have that

$$|F'_{\varepsilon_k}(w_k)[\varphi]|=|J'_{\varepsilon_k}(u_k)[\tilde{\varphi}_k]|\leq\sqrt{\delta_k}\,|||\tilde{\varphi}_k|||_{\varepsilon_k}\to 0 \text{ as } k\to\infty.$$

At this point to get the claim it is sufficient to show that

$$F'_{\varepsilon_k}(w_k)[\varphi]\to\left(E^+\right)'(w)[\varphi]. \tag{20}$$

In fact we have

$$\left|F'_{\varepsilon_k}(w_k)[\varphi]-\left(E^+\right)'(w)[\varphi]\right|\leq I_1+I_2+I_3,$$

where

$$\begin{aligned}I_1&=\int_{\mathrm{supp}\varphi}\left(\sum_{ij}g^{ij}_{q_k}(\varepsilon_k z)\frac{\partial w_k}{\partial z_i}(z)\frac{\partial \varphi}{\partial z_j}(z)\,|g_{q_k}(\varepsilon_k z)|^{1/2}-\delta_{ij}\frac{\partial w}{\partial z_i}(z)\frac{\partial \varphi}{\partial z_j}(z)\right)dz;\\I_2&=\int_{\mathrm{supp}\varphi}|\varphi(z)|\,|g_{q_k}(\varepsilon_k z)|^{1/2}\,|w_k(z)-w(z)|dz;\\I_3&=\int_{\mathrm{supp}\varphi}|\varphi(z)|\,|g_{q_k}(\varepsilon_k z)|^{1/2}\,|\left(w_k^+(z)\right)^{p-1}-(w(z))^{p-1}|dz.\end{aligned}$$

Because suppφ is a compact set, $|g^{ij}_{q_k}(\varepsilon_k z)-\delta_{ij}|\leq c\varepsilon_k\,|z|^2$ and by Step 1 we get (20).

Step3: *The limit function w is a least energy solution of*

$$\begin{cases}-\Delta w+w=(w^+)^{p-1} & \text{in } \mathbb{R}^n_+;\\ \frac{\partial w}{\partial\nu}=0 & \text{for } y=(\bar{y},0);\end{cases}$$

We will show that $w \neq 0$. We are in the case $P_k \cap \partial M \neq \emptyset$. We can choose $T > 0$ such that

$$P_k \subset I_{q_k}(\varepsilon_k T, \varepsilon_k T) \text{ for } k \text{ large enough.}$$

where q_k is a point in P_k. By definition w_k and by Lemma 10 there exist a q_k such that, for k large enough

$$\begin{aligned}
||w_k^+||_{L^p(B_{n-1}(0,T)\times[0,T])} &= \int_{B_{n-1}(0,T)\times[0,T]} \left|\chi_R(\varepsilon_k|\bar{z}|)\chi_\rho(\varepsilon_k z_n) u_k^+\left(\psi_{q_k}^\partial(\varepsilon_k z)\right)\right|^p dz = \\
&= \frac{1}{\varepsilon_k^n}\int_{B_{n-1}(0,\varepsilon_k T)\times[0,\varepsilon_k T]} \left|u_k^+\left(\psi_{q_k}^\partial(y)\right)\right|^p dy \geq \\
&\geq \frac{c}{\varepsilon_k^n}\int_{B_{n-1}(0,\varepsilon_k T)\times[0,\varepsilon_k T]} \left|u_k^+\left(\psi_{q_k}^\partial(y)\right)\right|^p |g_{q_k}(y)|^{1/2} dy = \\
&\geq \frac{c}{\varepsilon_k^n}\int_{I_{q_k}(\varepsilon_k T,\varepsilon_k T)} \left|u_k^+\right|^p d\mu_g \geq c\gamma > 0.
\end{aligned}$$

Since w_k converge strongly to w in $L^p(B_{n-1}(0,T)\times[0,T])$, we have $w \neq 0$.

We now show that

$$\left(\frac{1}{2} - \frac{1}{p}\right)|w^+|_p^p \leq m_e^+.$$

Since $u_k \in \mathcal{N}_{\varepsilon_k} \cap J_{\varepsilon_k}^{m_e^+ + \delta_k}$, it holds

$$\begin{aligned}
\frac{m_e^+ + \delta_k}{\frac{1}{2} - \frac{1}{p}} &\geq \frac{1}{\frac{1}{2} - \frac{1}{p}} J_{\varepsilon_k}(u_k) = \frac{1}{\varepsilon_k^n}\int_M |u_k^+|^p d\mu_g \geq \\
&\geq \frac{1}{\varepsilon_k^n}\int_{B_{n-1}(q_k,R/2)\times[0,\rho/2]} |u_k^+(\psi_{q_k}^\partial(y))|^p |g_{q_k}(y)|^{1/2} dy = \\
&= \int_{B_{n-1}(q_k,R/2\varepsilon_k)\times[0,\rho/2\varepsilon_k]} |u_k^+(\psi_{q_k}^\partial(\varepsilon_k z))|^p |g_{q_k}(\varepsilon_k z)|^{1/2} dz.
\end{aligned}$$

We set

$$f_k(z) = u_k^+(\psi_{q_k}^\partial(\varepsilon_k z))|g_{q_k}(\varepsilon_k z)|^{1/2}\zeta_k(z)$$

where ζ_k is the characteristic function of the set $B_{n-1}(q_k, R/\varepsilon_k) \times [0, \rho/\varepsilon_k]$. The sequence of function f_k is bounded in $L^p(\mathbb{R}_+^n)$, hence, up to subsequence, converges weakly to some $f \in L^p(\mathbb{R}_+^n)$. We get, for any $\varphi \in C_0^\infty(\mathbb{R}_+^n)$,

$$\int_{\mathbb{R}_+^n} f_k(z)\varphi(z)dz \to \int_{\mathbb{R}_+^n} w^+(z)\varphi(z)dz \text{ as } k \to \infty.$$

Hence f is equal to the positive function $w^+ = w \neq 0$. Moreover we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)|w|_p^p \leq \liminf_{k\to\infty}\left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}_+^n} |f_k(z)|^p dz \leq m_e^+.$$

Concluding $w \in \mathcal{N}^+$ and $E^+(w) \le m_e^+$, so w is a least energy solution.

Conclusion of the first case: *At this point we can show that, for any $T > 0$, it holds, for k large enough,*

$$\left(\frac{1}{2} - \frac{1}{p}\right) |w_k|^p_{L^p(B_{n-1}(0,T)\times[0,T])} \le \frac{2}{3}(1-\eta)m_e^+.$$

In fact we recall that for any $q \in \partial M$ the Riemannian metric $g_q(y)$ read through the Fermi coordinates is such that $g_q(\varepsilon_k z) = 1 + O(\varepsilon_k |z|)$. Hence fixed T

$$|g_q(\varepsilon_k z)|^{-1/2} \le \frac{2}{3} \text{ for } k \text{ big enough and for } z \in B_{n-1}(0,T) \times [0,T]$$

By this fact, using the definition of w_k and (15) we have, for k large,

$$\begin{aligned}|w_k^+|^p_{L^p(B_{n-1}(0,T)\times[0,T])} &\le \int_{B_{n-1}(0,T)\times[0,T]} |u_k^+(\psi^\partial_{q_k}(\varepsilon_k z))|^p |g_{q_k}(\varepsilon_k z)|^{1/2}\frac{2}{3}dz = \\ = \frac{2}{3}\frac{1}{\varepsilon_k^n}\int_{I(q_k,\varepsilon_k T,\varepsilon_k T)} |u_k^+|^p d\mu_g &\le \frac{2}{3}(1-\eta)\frac{m_e^+}{\left(\frac{1}{2}-\frac{1}{p}\right)}. \end{aligned} \tag{21}$$

On the other side by Step 3 we have that

$$E^+(w) = \left(\frac{1}{2} - \frac{1}{p}\right)|w|_p^p = m_e^+.$$

Now, by Step 1 there exists $T > 0$ such that, for k big enough we have

$$\frac{2}{3}(1-\eta)\frac{m_e^+}{\left(\frac{1}{2}-\frac{1}{p}\right)} < |w_k^+|^p_{L^p(B_{n-1}(0,T)\times[0,T])}. \tag{22}$$

By (21) and by (22) we have a contradiction.

The case $P_k^\varepsilon \cap \partial M = \emptyset$. we choose a point q_k interior to P_k^ε and we consider the normal coordinates at q_k . We set $w_k(z)$ as

$$u_k(x)\chi_R(\exp_{q_k}^{-1}(x)) = u_k(\exp_{q_k}(y))\chi_R(y) = u_k(\exp_{q_k}(\varepsilon_k z))\chi_R(\varepsilon_k z) = w_k(z).$$

Then $w_k \in H_0^1(B(0, R/\varepsilon_k)) \subset H^1(\mathbb{R}^n)$. Arguing as in the previous step, we can establish some properties of the function w_k. We omit the proof of single steps.

Step 1: w_k is bounded in H^1 and converge to some $w \in H^1$ weakly L^p_{loc} in and strongly in H^1.

Step 2: w is a weak solution of $-\Delta w + w = (w^+)^{p-1}$ in $\mathbb{R}^n$

STEP 3: w is strictly positive, and it is a least energy solution of $-\Delta w + w = |w|^{p-1}w$, that is

$$\left(\frac{1}{2}-\frac{1}{p}\right)|w|_p^p = E(w) = m_e = 2m_e^+. \tag{23}$$

CONCLUSION OF THE SECOND CASE: By (23) and (15) we have the contradiction

This concludes the proof. □

Remark 13. We point out that in the proof of Proposition 12, by Remark 9 and by (13) we showed that

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_e^+.$$

6 The map β

For any $u \in \mathcal{N}_\varepsilon$ we can define its center of mass as a point $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) = \frac{\displaystyle\int_M x|u^+(x)|^p d\mu_g}{\displaystyle\int_M |u^+(x)|^p d\mu_g}. \tag{24}$$

The application is well defined on $\mathcal{N}_\varepsilon$, since $u \in \mathcal{N}_\varepsilon$ implies $u^+ \neq 0$. In the following we will show that if $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ then $\beta(u) \in (\partial M)_{3\rho}$,using the concentration property (Prop. 12) of the function $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ if ε and δ are sufficiently small.

Proposition 14. *For any $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$, with ε and δ small enough, it holds*

$$\beta(u) \in (\partial M)_{3\rho}$$

Proof. Since $m_\varepsilon \to m_e^+$ and by Proposition 12 we get that for any $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ there exists $q \in \partial M$ such that

$$(1-\eta)m_e^+ \le \left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon^n}|u^+|^p_{L^p(I_q(\rho,R))}. \tag{25}$$

Since $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$ we have

$$\left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\varepsilon^n}|u^+|^p_{p,g} < m_e^+ + \delta. \tag{26}$$

Then by (25) and (26) we get

$$\int_{I_q(\rho,R)} \frac{|u^+|^p}{|u^+|^p_{p,g}} d\mu_g \geq \frac{1-\eta}{1+\frac{\delta}{m_e^+}}.$$

By definition of β we have

$$\begin{aligned}|\beta(u)-q| &\leq \left|\int_{I_q(\rho,R)} (x-q)\frac{|u^+|^p}{|u^+|^p_{p,g}} d\mu_g\right| + \left|\int_{M\smallsetminus I_q(\rho,R)} (x-q)\frac{|u^+|^p}{|u^+|^p_{p,g}} d\mu_g\right| \leq \\ &\leq 2\rho + D\left(1-\frac{1-\eta}{1+\frac{\delta}{m_e^+}}\right),\end{aligned}$$

where D is the diameter of the manifold M as a subset of $\mathbb{R}^n$. Choosing η and δ small enough we get the claim. □

Proposition 15. *The composition*

$$\beta \circ \Phi_\varepsilon : \partial M \to (\partial M)_{3\rho} \subset \mathbb{R}^n$$

is well defined and homotopic to the identity of ∂M.

Proof. By Proposition 14 and 8 the map $\beta \circ \Phi_\varepsilon : \partial M \to (\partial M)_{\rho(\partial M)}$ is well defined.

To prove that $\beta \circ \Phi_\varepsilon : \partial M \to (\partial M)_{3\rho}$ is homotopic to the identity it is enough to evaluate the map

$$\begin{aligned}\beta(\Phi_\varepsilon(q)) - q &= \frac{\int_{B_{n-1}(0,R)\times[0,\rho]} y|V_\varepsilon(y)\chi_R(|\bar{y}|)\chi_\rho(y_n)|^p dy}{\int_{B_{n-1}(0,R)\times[0,\rho]} |V_\varepsilon(y)\chi_R(|\bar{y}|)\chi_\rho(y_n)|^p dy} = \\ &= \frac{\varepsilon\int_{B_{n-1}(0,R/\varepsilon)\times[0,\rho/\varepsilon]} z|V(z)\chi_R(|\varepsilon\bar{z}|)\chi_\rho(\varepsilon z_n)|^p dz}{\int_{B_{n-1}(0,R/\varepsilon)\times[0,\rho/\varepsilon]} |V(z)\chi_R(|\varepsilon\bar{z}|)\chi_\rho(\varepsilon z_n)|^p dz}.\end{aligned}$$

By the exponential decay of V we get $|\beta(\Phi_\varepsilon(q)) - q| < c\varepsilon$, where c is a constant not depending on q. □

7 The set T_ε

To finish the proof of Theorem 4, it remains to show that there exists a critical point $\bar{u}$ of J_ε in $\mathcal{N}_\varepsilon$ with $m_e^+ + \delta < J_\varepsilon(\bar{u}) < c_\varepsilon$, for bounded constants c_ε. As explained in Section 3, this is achieved by constructing a set T_ε which contains only positive functions, is contractible in $\mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$ and contains

$\Phi_\varepsilon(\partial M)$. The process of building the set T_ε is analogous to the process of section 6 of [4]; for clearness we prefer to show it.

To define the set T_ε we use the functions $Z_{\varepsilon,q}(x)$ as defined in (7). We recall that $Z_{\varepsilon,q}(x) \in H^1_g(M)$ are positive functions. Let $W(x) \in H^1(\mathbb{R}^n_+)$ be any positive function and denote as usual $W_\varepsilon(x) = W\left(\frac{x}{\varepsilon}\right)$. For $q_0 \in \partial M$ a fixed point on the boundary of M we introduce the functions

$$v_\varepsilon(x) := \begin{cases} W_\varepsilon(y(\xi))\tilde{\chi}(y(\xi)) & \text{if } \xi \in I_{q_0}(R,\rho); \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

where $y(\xi) = \left(\psi^\partial_{q_0}\right)^{-1}$ and $\tilde{\chi}(y) = \chi_R(\bar{y})\chi_\rho(y_n)$ as in the previous part of the paper.

We define the cone

$$C_\varepsilon := \{u(x) := \theta v_\varepsilon(x) + (1-\theta) Z_{\varepsilon,q}(x) \ : \ \theta \in [0,1], q \in \partial M.\} \subset H^1_g(M). \tag{28}$$

By the properties of the map Φ_ε proved in Proposition 8, we have that C_ε is compact and contractible in $H^1_g(M)$. We now project it on the Nehari manifold $\mathcal{N}_\varepsilon$ by the factor $t_\varepsilon(u)$ to obtain

$$T_\varepsilon := \left\{ t_\varepsilon(u)u : u \in C_\varepsilon, t_\varepsilon^{p-2}(u) = \frac{|||u|||^2_\varepsilon}{\frac{1}{\varepsilon^n}|u|^p_{p,g}} \right\} \subset \mathcal{N}_\varepsilon. \tag{29}$$

We get that $\Phi_\varepsilon(\partial M) \subset T_\varepsilon$, that T_ε contains only positive functions and that it is compact and contractible in $\mathcal{N}_\varepsilon$. Hence if we define

$$c_\varepsilon := \max_{u \in C_\varepsilon} J_\varepsilon(t_\varepsilon(u)u)$$

we get that $T_\varepsilon \subset \mathcal{N}_\varepsilon \cap J_\varepsilon^{c_\varepsilon}$. The last step is to prove the following proposition.

Proposition 16. *There exists a constant $c > 0$ such that for ε small enough it holds $c_\varepsilon < c$.*

Proof. By the definition of the Nehari manifold, we recall that for $u \in C_\varepsilon$ it holds

$$J_\varepsilon(t_\varepsilon(u)u) = \left(\frac{1}{2} - \frac{1}{p}\right) t^2_\varepsilon(u)|||u|||^2_\varepsilon = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{|||u|||_\varepsilon^{\frac{2p}{p-2}}}{\left(\frac{1}{\varepsilon^n}|u|^p_{p,g}\right)^{\frac{2}{p-2}}}. \tag{30}$$

Arguing as (9), (10), (11) for v_ε and W instead of $Z_{\varepsilon,q}$ and V, we find that there exists a constant $k_1 > 0$ such that

$$|||u|||_\varepsilon^2 \leq ||W||_{H^1}^2 + ||V||_{H^1}^2 + k_1 \tag{31}$$

for ε small enough. Moreover, for ε small enough, we find constants $k_2 > 0$ and $k_3 > 0$ such that

$$\frac{1}{\varepsilon^n}|v_\varepsilon|_{p,g}^p \geq |W|_p^p - k_2 > 0, \qquad \frac{1}{\varepsilon^n}|Z_{\varepsilon,q}|_{p,g}^p \geq |V|_p^p - k_3 > 0.$$

Hence, since v_ε and $Z_{\varepsilon,q}$ are positive functions and $\theta \in [0,1]$, there exists k_4 such that

$$\frac{1}{\varepsilon^n}|u|_{p,g}^p \geq \frac{1}{\varepsilon^n}\max\{|\theta v_\varepsilon|_{p,g}^p, |(1-\theta)Z_{\varepsilon,q}|_{p,g}^p\} \geq k_4 \tag{32}$$

for ε small enough. Putting together (30), (31) and (32) we get the thesis. □

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